M-math (2019) Final Exam
Subject : Stochastic Processes

Time : 3.00 hours Max.Marks 50.

1. Let G be a bounded open set in $\mathbb{R}^d$ and $g : \partial G \to \mathbb{R}$ be a bounded measurable function. Here ∂G is the boundary of G. Let $H \subset G$ be an open set. Let τ_H be the first exit time from H of a Brownian motion (W_t). Define $W_{\tau_H} := W_t, t = \tau_H < \infty; = 0 \ if \ \tau_H = \infty$. Let $P_x, x \in \mathbb{R}^d$ be the law of Brownian motion starting at x.
a. Define $\mu(x, A) := P_x\{W_{\tau_H} \in A\}$. Show that if $x \in H$, the measure $\mu(x, \cdot)$ is supported on the boundary ∂H.
b. Let $h(x) := E_x(g(W_{\tau_G})I_{\{\tau_G<\infty\}})$. Show that $h(x) = \int_{\partial H} h(y)\mu(x, dy)$, for $x \in H$. (5+ 10)

2. Let $T = [0, \infty)$ and $\{\mu_{t_1,\cdots,t_k}; t_i \in T, i = 1, \cdots, k, k \geq 1\}$ be a consistent family of probability measures. Let $X_t : \mathbb{R}^T \to \mathbb{R}, X_t(x); = x(t)$. Suppose for every countable subset S of T, there exists probability measures P_S, on $\sigma\{X_t, t \in S\}$, satisfying $P_S \circ \pi^{-1}_{t_1,\cdots,t_k} = \mu_{t_1,\cdots,t_k}$ where $\pi_{t_1,\cdots,t_k}(x) := (x_{t_1}, \cdots, x_{t_k}), x \in \mathbb{R}^T$. Show that there exists a probability P on $(\mathbb{R}^T, \mathcal{R}^T)$ whose finite dimensional distributions are given by the family $\{\mu_{t_1,\cdots,t_k}\}$. (10)

3. Let (W_t) be standard one dimensional Brownian motion. Let $\mathcal{F}_{0+} := \bigcap_{\epsilon>0} \mathcal{F}_\epsilon, \mathcal{F}_\epsilon := \sigma\{W_t, 0 < t < \epsilon\}$. Show that if $A \in \mathcal{F}_{0+}$ then $P(A) = 0 \ or \ 1$. (12)

4. Let (W_t) be a standard one dimensional Brownian motion and and for $\alpha > 0$, let $\tau_\alpha := \inf\{t > 0 : W_t \geq \alpha\}$. Use an appropriate martingale, to show that $Ee^{-t\tau_\alpha} = e^{-\alpha\sqrt{2t}}, t > 0$. (13)

5. Let (W_t) as in 3). Show that for every $x \in \mathbb{R}, x \neq 0$, almost surely, the set $Z_x := \{t : W_t = x\}$ is closed, unbounded, nowhere dense, perfect set of Lebesgue measure zero. (5)